Lie Superalgebras

Generators of the superconformal group belong to
lie super algebra (LSA)

This is an algebra A with a $\mathbb{Z}_2 = \mathbb{Z}/2$ grading

$$A = \bigoplus_{\alpha \in \mathbb{Z}_2} A_\alpha = A_{\bar{0}} \oplus A_{\bar{1}}$$

such that $A_\alpha A_\beta \subseteq A_{\alpha+\beta}$

elements of $A_{\bar{0}}$ are called "even", those of $A_{\bar{1}}$ "odd"

Define commutator: $[a,b] = ab - (-1)^{\deg(a)\deg(b)} ba$

where $a, b \in A$ and $\deg \in \mathbb{Z}_2$

A "Lie superalgebra" is a superalgebra $G = G_{\bar{0}} \oplus G_{\bar{1}}$
with operation $[\, , \}$ satisfying

$$[a,b\} = -(-1)^{(\deg a)(\deg b)} [b,a]$$

$$[a, [b,c]] = [[a,b],c] + (-1)^{(\deg a)(\deg b)} [b, [a,c]]$$

(Jacobi identity)

we have even $\underset{[\,,\,\}}{\times}$ even = even, odd $\times$ odd = even,

odd $\times$ even = odd, even $\times$ odd = odd

<u>construction</u>:

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space

$\longrightarrow$ End$V$ is endowed with $\mathbb{Z}_2$ grading

$\rightarrow$ associative superalgebra

notation: $\mathrm{End}(V) = \ell(V) = \ell(m,n)$, where $m = \dim V_{\bar{0}}$, $n = \dim V_{\bar{1}}$

$$\ell(V) = \underset{\underset{G_0}{\shortparallel}}{\ell(V)_0} \oplus \underset{\underset{G_1}{\shortparallel}}{\ell(V)_1}$$

We can further decompose $\ell(V)_1 = \ell_{-\bar{1}} + \ell(V)_{\bar{0}} + \ell_{\bar{1}}$

Let $e_1, \dots, e_m, e_{m+1}, \dots, e_{m+n}$ be a basis of $V = V_{\bar{0}} \oplus V_{\bar{1}}$

$a \in \ell(V)$ can be written as: $a = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$,

where α is $(m \times m)$-matrix

δ is $(n \times n)$- "

β is $(m \times n)$- "

γ is $(n \times m)$- "

even elements have the form $\begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}$,

odd elements have the form $\begin{bmatrix} 0 & \beta \\ \gamma & 0 \end{bmatrix}$

$\rightarrow \ell_{\bar{1}} = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}$ and $\ell_{-\bar{1}} = \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix}$

$$\Rightarrow \Big[\underbrace{\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}}_{\text{even}}, \underbrace{\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}}_{\text{odd}}\Big] = \underbrace{\begin{pmatrix} 0 & \alpha\beta - \beta\delta \\ 0 & 0 \end{pmatrix}}_{\text{odd}}$$

$$\Big[\underbrace{\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}}_{\text{even}}, \underbrace{\begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}}_{\text{odd}}\Big] = \underbrace{\begin{pmatrix} 0 & 0 \\ \delta\gamma - \gamma\alpha & 0 \end{pmatrix}}_{\text{odd}}$$

$$\Big[\underbrace{\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}}_{\text{odd}}, \underbrace{\begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}}_{\text{odd}}\Big] = \underbrace{\begin{pmatrix} \beta\gamma & 0 \\ 0 & \gamma\beta \end{pmatrix}}_{\text{even}}$$

We call $\ell_{\bar{1}}$ and $\ell_{-\bar{1}}$ $\ell_{\bar{0}}$-modules

(form particular rep. under $\ell_{\bar{0}}$)

→ type 1 representation : $G_1 = G_{\bar{1}} \oplus G_{-\bar{1}}$

We can also have a type 2 rep. where G_1 is irreducible, i.e one cannot set β or γ to zero and odd elements are of the form $\begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$

type 2 classification:

$G = G_0 + G_1$	G_0	G_0 Rep on G_1
$B(m,n)$	$B_m + C_n$ with α, β	vector × vector
$D(m,n)$	$D_m + C_n$ with α, β	vector × vector
$D(2,1,\alpha)$	$A_1 + A_1 + A_1$	vector × vector × vector
$F(4)$	$B_3 + A_1$	spinor × vector
$G(3)$	$G_2 + A_1$	spinor × vector
$Q(n)$	A_n	adjoint

convention: B_m is Lie-algebra of $SO(2m+1)$, C_n is Lie algebra of $Sp(2n)$

type 1 classification:

G_0 acts on $G_{\bar{1}}$ and $G_{-\bar{1}}$ as irreducible representations which are contragradient (weights$(G_{-\bar{1}})$ = − weights$(G_{\bar{1}})$)

$G = G_0 + G_1$	G_0	G_0 rep on $G_{-\bar{1}}$
$A(m,n)$	$A_m + A_n + C$	vector × vector × C
$A(m,m)$	$A_m + A_m$	vector × vector
$C(n)$	$C_{n-1} + C$	vector × C

matrix construction:

- $sl(m,n)$:

$(m+n, m+n)$-matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ of zero supertrace

$\mathrm{str} = \mathrm{tr}\,\alpha - \mathrm{tr}\,\delta$

- $osp(m,n)$:

Define bi-linear form F with $F(x,y) = 0$ for $x \in V_{\bar{0}}$, $y \in V_{\bar{1}}$, $F(x,y) = F(y,x)$ for $x,y \in V_{\bar{0}}$ and $F(x,y) = -F(y,x)$ for $x,y \in V_{\bar{1}}$

then $osp(m,n) = \{ a \in l(m,n)_s \mid F(a(x),y) = -(-1)^{s \deg x} F(x, a(y)) \}$

$s \in \mathbb{Z}_2$, $\deg(a)$

Then one has:

$$A(m,n) = sl(m+1, n+1) \quad (m \neq n,\ m,n \geq 0)$$

$$A(m,m) = sl(m+1, m+1)/I \quad (m \geq 0)$$

$$B(m,n) = osp(2m+1, 2n) \quad (m \geq 0\ \ n > 0)$$

$$C(n) = osp(2, 2n) \quad (n > 0)$$

$$D(m,n) = osp(2m, 2n) \quad (m \geq 2, n \geq 0)$$

(There are also algebras $Q(n)$ and $P(n)$ on which we do not elaborate)

Let's give an explicit description for the case $osp(2m+1, 2n)$:

$$F = \left(\begin{array}{ccc|cc} 0 & \mathbb{1}_m & 0 & & \\ \mathbb{1}_m & 0 & 0 & & \\ 0 & 0 & 1 & & \\ \hline & & & 0 & \mathbb{1}_n \\ & & & -\mathbb{1}_n & 0 \end{array}\right)$$

$\Rightarrow$ $a \in osp(m,n)$ is of the form

$$\left(\begin{array}{ccc|cc} a & b & u & x & x_1 \\ c & -a^T & v & y & y_1 \\ -v^T & -u^T & 0 & z & z_1 \\ \hline y_1^T & x_1^T & z_1^T & d & e \\ -y^T & -x^T & -z^T & f & -d^T \end{array}\right)$$

$\rightarrow$ G_1 is irreducible rep. of G_0

For $osp(2m, 2n)$ we get

$$F = \left(\begin{array}{cc|cc} 0 & \mathbb{1}_m & & \\ \mathbb{1}_m & 0 & & \\ \hline & & 0 & \mathbb{1}_n \\ & & -\mathbb{1}_n & 0 \end{array}\right)$$

$\Rightarrow$ a is of same form as above with central row and column deleted

How do SCA's fit into this classification?

We are searching superalgebras whose even part G_0 contains conformal group $SO(d,2)$

G_1 should be a spinor representation of G_0

$B(m,n)$ and $D(m,n)$ have SO-subalgebras <u>but</u> in vector-rep.

$F(4)$ has subalgebra $B_3 = SO(7)$, represented as spinor!

$\Rightarrow$ $F(4)$ is superconformal algebra!

R-symmetry: $A_1 = SU(2) = SO(3)$

$\rightarrow$ $F(4)$ is the $d=5$, $n=1$ superconformal algebra

Now, recall $\underset{\text{spinor}}{SO(5)} = \underset{\text{vector}}{Sp(2)} = C_2$

$\rightarrow$ $B(m,2)$ and $D(m,2)$ are superconformal algebras in $d=3$ with R-symmetry $SO(2m+1)$ and $SO(2m)$

$SO(6) = SU(4)$

spinor = vector

$\rightarrow$ $A(3,m)$ is SCA in $d=4$ with R-symmetry $A_m + C_1$ i.e $SU(m+1) \times U(1) = U(m+1)$

$SO(8)$ admits triality

→ spinor representation is equivalent to vector representation!

⇒ $B(4,n)$ is superconformal algebra in $d=6$ with R-symmetry $C_n = Sp(n)$

same trick does <u>not</u> work for $SO(m)$ with $m>8$

⇒ No superconformal algebras in $d>6$!!